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LETTER TO THE EDITOR

Geometric phase, geometric distance and length of the curve in quantum evolution

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Abstract. The geometric phase and the geometric distance function are intimately related via length of the curve (a concept we introduce) for any parametric evolution of the quantum system. We offer an interpretation of the non-adiabatic Berry phase as the integral over a difference between the differentials of two geometric quantities, which enables us to say that the geometric phase is just (half) the integral of the contracted length of the curve that the system traverses during a cyclic excursion.

With the inclusion of the geometric phase into the general context of quantum mechanics, our understanding has been changed in a dramatic way. We now wish to view quantum theory as a purely geometric theory. After Berry's discovery [1] of a new phase factor accompanying adiabatic evolution in quantal adiabatic theorem, Simon [2] interpreted this mysterious phase as a result of holonomy in a line bundle over parameter space on which the system's Hamiltonian depends. The next generalization was for non-adiabatic evolutions of the quantum system, which was done by Aharonov and Anandan [3]. They studied the case of the phase acquired by a system's wavefunction upon cyclic evolution in the projective Hilbert space. Further theoretical studies of this geometric phase were taken up by Samuel and Bhandari [4] in a more general context. The Berry phase made its appearance (to name a few) in molecular dynamics in the Born-Oppenheimer approximation [5], the quantum Hall effect [6], the case of a Hamiltonian description of quantum field theory that develops anomalies [7] and, recently, in the case of a relativistic Dirac Hamiltonian with Thomas precession and spin-orbit interactions [8].

A less well known geometric quantity is the 'distance' between two quantum states. Provost and Valle [9] were the first to realize the necessity of introducing the geometric structures on the Hilbert space of quantum states. Their idea, in turn, came from nuclear physics [10], where the Riemannian structure has been introduced to describe the collective behaviours of the nucleons. They have shown how the Hermitian product on the projective Hilbert space induces a meaningful metric tensor on any manifold of quantum states. The physical significance of the metric structure is that they are related to the dispersion of the quantum operators acting on the Hilbert space that generates the evolution. There is revival of interest in these ideas due to the recent work of Anandan and Aharonov [11] and independently by Montgomery [12]. In the case of the time evolution of a quantum system, the distance between quantum states along a given curve $\hat{C}$ in the projective Hilbert space $\mathscr{P}$ is the time integral of the

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uncertainty of energy and is geometric in the sense that this distance function is independent of the particular Hamiltonian used to evolve a quantum system along a given curve in $\mathcal{P}$. It is also independent of the phases of two infinitesimally nearby states and, therefore, depends only on the points in $\mathcal{P}$ to which they project. Because of this property, even if we change the Hamiltonian by some amount, it will give only a different phase factor in the state vector and the distance function will be unaltered. We have shown [13] that this fact is, in turn, a consequence of invariance of the (infinitesimal) 2-point Bargmann invariant [14] under U(1) action. On our line of query, we asked whether the geometric distance function is related to some other physical quantity during the evolution. We found that time average of the acceleration of a quantum particle is related to the distance function during the period of the evolution [15]. Since acceleration is related to the distance function in some way, the external force to which the particle is subjected is decided by the quantum metric tensor. It is a very important result in the sense that, three hundred years after Newton, we now understand that the acceleration of a quantum system is a geometric quantity related to the evolution of the system.

Another geometric quantity of interest that we have recently recognized is referred to as the 'length of the curve' along which the quantum system is traversed [16]. It is the property of the whole curve $\hat{C}$ in $\mathcal{P}$. When the system obeys the Schrödinger time evolution equation then the length of the curve is a t-invariant quantity. So the length of the curve is 'as geometric as' the geometric phase. For instance [16] we have found that in the case of a spin-$\frac{1}{2}$ particle undergoing precession in a homogeneous magnetic field, the total length of the curve in one period is just equal to the square root of π times the total solid angle subtended by the orbit of motion in a sphere of unit radius. We emphasize here that the length of the curve during a cyclic evolution could be measured in similar experimental set-ups that were used to reveal the existence of the non-adiabatic Berry phase, which may reflect the geometric property of the motion of the system in $\mathcal{P}$. With this recognization, it is clear that the projective Hilbert space is enriched with geometric objects like geometric phase, distance and length of the curve. Our motive in this letter is three-fold; (i) to show that the geometric phase and the geometric distance function are intimately related through the length of the curve (not necessarily in the case of time evolution but for any parametric evolution), (ii) to give a new interpretation to the non-adiabatic Berry phase for cyclic evolution and possibly for the non-Hermitian generator of the evolution and (iii) to provide an alternative, tractable algorithm for calculating the geometric phase using geometric concepts like distance and length in quantum evolution.

For completeness we define here the geometric phase, distance and length of the curve. Consider $\{\psi(\lambda)\}$ be a set of normalized vectors belonging to a Hilbert space $\mathcal{H}$ of dimension $N+1$. Then we can define a projective Hilbert space of dimension one less, i.e. N. It consists of a set of rays of the Hilbert space $\mathcal{H}$, where the rays are defined as the equivalence classes of states differing only in phase. The equivalence relation is $|\psi\rangle \sim |\psi'\rangle$ if $|\psi'\rangle = c|\psi\rangle$ where $0 \neq c \in \mathbb{C}^*$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$ is a multiplicative group of non-zero complex numbers. The projective Hilbert space is $\mathcal{P} = \mathcal{P}_N(\mathbb{C}) = \{\mathcal{H} - \{0\}\}/\mathbb{C}^*$, $\mathcal{P}_N(\mathbb{C})$ is the quantum state space. All vectors in $\mathcal{H}$ are projected on to $\mathcal{P}_N(\mathbb{C})$ and the physical states are elements of $\mathcal{P}$ and represented as points in $\mathcal{P}$. There is also a structure order in $\mathcal{P}$, namely two state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ define a 'line' in $\mathcal{P}$, and a third state $|\psi_3\rangle$ may or may not be in this line. In latter case this shows the non-transitivity and this property is responsible for geometric phase for a cyclic evolution of states.

First we define the geometric phase, following the prescription of Anandan [17]. Let $\mathscr{L}$ be the natural line bundle over $\mathscr{P}$ with each fibre being the ray of vectors in the point to which it projects under the projection map Π: $\mathscr{L} \to \mathscr{P}$. Let $|\psi(\lambda)\rangle$ be a curve in $\mathscr{L}$ which projects to a curve $\hat{C}$ in $\mathscr{P}$. Then $\mathrm{d}/\mathrm{d}\lambda|\psi(\lambda)\rangle$ is the tangent vector to the curve $|\psi(\lambda)\rangle$ and $\langle\psi|\mathrm{d}/\mathrm{d}\lambda|\psi\rangle$ describes the parallel transport of the state $|\psi\rangle$ with respect to the connection defined on it. It was shown that if $\hat{C}:[0,\Lambda]\to\mathscr{P}$ is a closed curve, then $|\psi(\Lambda)\rangle = \mathrm{e}^{\mathrm{i}\Phi}|\psi(0)\rangle$ where Φ is the total phase and in general complex. If $|\psi(\lambda)\rangle$ obeys the parameter evolution equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\lambda}|\psi(\lambda)\rangle = A(\lambda)|\psi(\lambda)\rangle \tag{1}$$

then choose a section of the curve $|\tilde{\psi}\rangle$ as $|\tilde{\psi}(\lambda)\rangle = \exp(-\mathrm{i}f(\lambda))|\psi(\lambda)\rangle$ with $\langle\tilde{\psi}|\tilde{\psi}\rangle = 1$ and $f(\lambda)$ is any smooth complex function satisfying $f(\Lambda)-f(0)=\Phi$. Then $|\tilde{\psi}(\lambda)\rangle$ is a single valued state, i.e. $|\tilde{\psi}(\Lambda)\rangle = |\tilde{\psi}(0)\rangle$. The single valued states are important because they only depend on the image of the evolution in $\mathscr{P}$. Therefore these states are used to define the geometric phase. If Φ is the total phase, then Anandan has shown that

$$\Phi = \beta - \int_0^\Lambda \langle\psi(\lambda)|A(\lambda)|\psi(\lambda)\rangle\,\mathrm{d}\lambda$$

and

$$\beta = \mathrm{i}\int_0^\Lambda \left\langle \tilde{\psi}(\lambda)\middle|\frac{\mathrm{d}}{\mathrm{d}\lambda}\tilde{\psi}(\lambda)\right\rangle \mathrm{d}\lambda = \mathrm{i}\oint\langle\tilde{\psi}|\mathrm{d}\tilde{\psi}\rangle. \tag{2}$$

Here $\mathrm{i}\langle\tilde{\psi}(\lambda)|\mathrm{d}|\tilde{\psi}(\lambda)\rangle$ is the 'connection form' analogous to the vector potential A in electromagnetic field that represents a connection form over spacetime. The normalization of $|\tilde{\psi}\rangle$ ensures that it is real. Thus β is the geometric phase for an arbitrary cyclic transformation. It is a part of the total phase Φ acquired by the state vector during the cyclic evolution which is geometric in nature and is independent of the generator used to cause the motion. That is to say the geometric phase is independent of the generator normalization. Because by changing the generator of the parametric evolution by some fixed amount, we will have a state vector whose overall phase is different but the extra phase will be shifted to the dynamical part in such a way that it will not affect the single-valued state $|\tilde{\psi}\rangle$ and hence the geometric phase. It only depends on the motion of the system given by the closed curve $\hat{C}$ in the set of rays in $\mathscr{P}$.

Now we briefly discuss the second geometric object in $\mathscr{P}$. The distance function between two infinitesimal close quantum states is a geometric quantity for all quantum evolution. To see this let us consider two non-orthogonal quantum states $|\psi_1\rangle$, $|\psi_2\rangle$ and their respective projections $\Pi(|\psi_1\rangle)$, $\Pi(|\psi_2\rangle)$ onto the projective Hilbert space $\mathscr{P}$. Then one can define the 'Bargmann angle' [18] between two rays as

$$R(\psi_1,\psi_2) = \frac{|\langle\psi_1|\psi_2\rangle|^2}{\|\psi_1\|^2\|\psi_2\|^2} = \cos^2\theta \tag{3}$$

with $0 \leqslant R \leqslant 1$.

The Bargmann angle describes the interference between two states. Anandan and Aharonov [11] have used similar relation and described the transition probability between two states in a geometric way. They concluded that the probability of transition $|\langle\psi_1|\psi_2\rangle|^2$ between two states satisfy (3) (with $\|\psi_1\|^2$, $\|\psi_2\|^2 = 1$), where θ is half the distance along any geodesic joining $\Pi(|\psi_1\rangle)$ and $\Pi(|\psi_2\rangle)$. If we allow $\Pi(|\psi_1\rangle)$ and $\Pi(|\psi_2\rangle)$

to be separated by an infinitesimal distance then we have the Fubini-Study metric, given by

$$ds^2 = 4(1 - |\langle\psi_1|\psi_2\rangle|^2). \tag{4}$$

This is obtained by using (3) and letting $\theta = ds/2$. Thus the inner product in $\mathscr{H}$ gives a metric in $\mathscr{P}$. For a fibre bundle description of these concepts see [11, 19]. Another way of obtaining the Fubini-Study metric on $\mathscr{P}$ is by taking the inner product of the horizontal component of the tangent vector in $\mathscr{H}$ and multiplying it by $4\, d\lambda^2$. That is, we can decompose the tangent vector into unique horizontal and vertical vectors as follows:

$$\frac{d}{d\lambda}|\psi(\lambda)\rangle = \frac{\delta}{d\lambda}|\psi(\lambda)\rangle + \frac{D}{d\lambda}|\psi(\lambda)\rangle \tag{5}$$

where

$$\frac{\delta}{d\lambda}|\psi(\lambda)\rangle \equiv \frac{d}{d\lambda}|\psi(\lambda)\rangle - \frac{\langle\psi(\lambda)|(d/d\lambda)|\psi(\lambda)\rangle}{\langle\psi(\lambda)|\psi(\lambda)\rangle}|\psi(\lambda)\rangle$$

and

$$\frac{D}{d\lambda}|\psi(\lambda)\rangle \equiv \frac{\langle\psi(\lambda)|(d/d\lambda)|\psi(\lambda)\rangle}{\langle\psi(\lambda)|\psi(\lambda)\rangle}|\psi(\lambda)\rangle.$$

Taking into account the evolution equation (1) we can show that

$$\Delta A^2(\lambda) = \frac{1}{\langle\psi|\psi\rangle}\left\langle\frac{\delta\psi}{d\lambda}\middle|\frac{\delta\psi}{d\lambda}\right\rangle = \left(\frac{1}{2}\frac{ds}{d\lambda}\right)^2 \tag{6}$$

where

$$\frac{1}{\langle\psi|\psi\rangle^{1/2}}\frac{\delta}{d\lambda}|\psi(\lambda)\rangle$$

is the tangent vector to the curve $\hat{C}$ in $\mathscr{P}$ and ΔA is a non-negative number that satisfies [20]

$$\Delta A^2(\lambda) = \frac{1}{\langle\psi|\psi\rangle}\langle\psi|\left(A - \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle}\right)^+\left(A - \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle}\right)|\psi\rangle.$$

Hence

$$ds^2 = 4\Delta A^2(\lambda)\, d\lambda^2 \qquad s = 2\int \Delta A(\lambda)\, d\lambda \tag{7}$$

where s is the distance along curve $\hat{C}$ as measured by the Fubini-Study metric. This is a geometric quantity in the sense that there would be an infinite number of $A(\lambda)$'s generating a given motion in $\mathscr{P}$ with same ΔA. Thus the Fubini-Study metric is fixed by the non-negative number $\Delta A(\lambda)$, implying thereby that the geometry of the state space is fixed by $\Delta A(\lambda)$; consequently $A(\lambda)$ alone cannot change the geometry of the projective Hilbert space $\mathscr{P}$.

Another important geometric quantity is the 'length of the curve' along which the quantum system is transported. It is possible to define this for arbitrary evolutions in the same vein as the geometric distance function. On a proper Riemannian manifold the presence of a metric allows the definition of the length $l(C)$ of a differentiable curve in $\mathscr{P}$, which is traced out by the state vector $|\tilde{\psi}(\lambda)\rangle$. Below, we define it.

Let ψ be a curve $C:[0,\Lambda]\to\mathscr{H}$. Then choose a section of the curve $\tilde{\psi}(\lambda)$ which is differentiable along C such that the length of the curve $\tilde{\psi}(\lambda)$ along which the system evolves from point $\tilde{\psi}(0)$ to a point $\tilde{\psi}(\Lambda)$ (or from a parameter value $\lambda=0$ to $\lambda=\Lambda$) is a number defined as

$$l(c)=l(\tilde{\psi})|_0^\Lambda=2\int_0^\Lambda\left\langle\frac{\mathrm{d}\tilde{\psi}(\lambda)}{\mathrm{d}\lambda}\middle|\frac{\mathrm{d}\tilde{\psi}(\lambda)}{\mathrm{d}\lambda}\right\rangle^{1/2}\mathrm{d}\lambda \tag{8}$$

where $|\mathrm{d}\tilde{\psi}(\lambda)/\mathrm{d}\lambda\rangle$ is the velocity vector in the projective Hilbert space $\mathscr{P}$ of the curve $\tilde{\psi}(\lambda)$ at point λ along the path of evolution of the state vector. It also tangent vector to the curve $\tilde{\psi}(\lambda)$.

We would like to spell out some of the general properties of the length of the curve. First of all the integral (8) exist in the interval $[0,1]$, since the integrand is continuous. The length of a broken C curve is defined as the (finite) sum of the length of its C pieces. All curves deduced from C by a change of parameter λ to λ' with $\tilde{\psi}$: $\lambda'\to\lambda=\tilde{\psi}(\lambda')$ have the same length. Therefore the number $l(C)$ is independent of the parametrization of its image set, i.e. for a smooth transformation from parameter λ to λ' with $\mathrm{d}\lambda/\mathrm{d}\lambda'>0$ the length of the curve remains unaltered. Hence the length is a property of the geometrical curve, defined by the equivalence class of parametrized paths and is λ-invariant.

From these considerations we can define an infinitesimal length of the curve during an infinitesimal change in the parameter (for an arbitrary parametric evolution of the state) as

$$\mathrm{d}l=2\left\langle\frac{\mathrm{d}\tilde{\psi}(\lambda)}{\mathrm{d}\lambda}\middle|\frac{\mathrm{d}\tilde{\psi}(\lambda)}{\mathrm{d}\lambda}\right\rangle^{1/2}\mathrm{d}\lambda. \tag{9}$$

If the parameter λ is such that the quantity $\langle(\mathrm{d}\tilde{\psi}(\lambda)/\mathrm{d}\lambda)|(\mathrm{d}\tilde{\psi}(\lambda)/\mathrm{d}\lambda)\rangle^{1/2}$ is constant along C, the curve is said to be parametrized proportionally to the arc length. The quantity $\mathrm{d}l/\mathrm{d}\lambda=u_{\mathscr{P}}$ is called the magnitude of the rate of change of (with respect to λ) arc length of the curve C. Therefore for a proportionally parametrized curve the quantity $u_{\mathscr{P}}$ is constant during the evolution.

We now proceed to show that the geometric phase and distance are related via length of the curve for an arbitrary parametric evolution of the quantum system. In fact, we will prove that the geometric phase for an arbitrary cyclic evolution of a quantum state is half of the integral of the contracted length of the curve $\hat{C}$ in $\mathscr{P}$ along which the system moves. This is easily done by evaluating the quantity $\mathrm{d}s^2$, the square of the infinitesimal Fubini-Study distance. By definition $\Delta A^2(\lambda)$ is

$$\Delta A^2(\lambda)=\frac{1}{\langle\psi(\lambda)|\psi(\lambda)\rangle}\left\langle\frac{\delta\psi(\lambda)}{\mathrm{d}\lambda}\middle|\frac{\delta\psi(\lambda)}{\mathrm{d}\lambda}\right\rangle.$$

Making use of the modified state vector $|\tilde{\psi}\rangle$ we have

$$\left\langle\psi(\lambda)\middle|\frac{\mathrm{d}\psi(\lambda)}{\mathrm{d}\lambda}\right\rangle=\mathrm{i}\frac{\mathrm{d}f}{\mathrm{d}\lambda}+\left\langle\tilde{\psi}(\lambda)\middle|\frac{\mathrm{d}}{\mathrm{d}\lambda}\tilde{\psi}(\lambda)\right\rangle. \tag{10}$$

Hence

$$\frac{\delta}{\mathrm{d}\lambda}|\psi(\lambda)\rangle=\mathrm{e}^{\mathrm{i}f(\lambda)}\left(\left|\frac{\mathrm{d}\tilde{\psi}(\lambda)}{\mathrm{d}\lambda}\right\rangle-\left\langle\tilde{\psi}(\lambda)\middle|\frac{\mathrm{d}}{\mathrm{d}\lambda}\tilde{\psi}(\lambda)\right\rangle|\tilde{\psi}(\lambda)\rangle\right). \tag{11}$$

Therefore

$$\Delta A^2(\lambda)=\left\langle\frac{\mathrm{d}\tilde{\psi}(\lambda)}{\mathrm{d}\lambda}\middle|\frac{\mathrm{d}\tilde{\psi}(\lambda)}{\mathrm{d}\lambda}\right\rangle-\left(\mathrm{i}\left\langle\tilde{\psi}(\lambda)\middle|\frac{\mathrm{d}\tilde{\psi}(\lambda)}{\mathrm{d}\lambda}\right\rangle\right)^2. \tag{12}$$

This leads to the expression

$$\mathrm{d}l^2-\mathrm{d}s^2=\left(2\mathrm{i}\left\langle\tilde{\psi}(\lambda)\middle|\frac{\mathrm{d}}{\mathrm{d}\lambda}\middle|\tilde{\psi}(\lambda)\right\rangle\right)^2\mathrm{d}\lambda^2. \tag{13}$$

The expression (13) is the relation between the geometric phase and the distance for any parametric evolution via length of the curve. If the parameter happens to be the time and A the Hamiltonian, the generator of the time translation then (13) gives the result obtained in the [16]. In proving this, we have not assumed that A should be Hermitian and linear, so (13) may hold good for non-Hermitian generators also. On writing (13) in a particular form we can see that it is nothing but (half) the integral of the contracted length of the curve. Since

$$\beta=\tfrac{1}{2}\int_0^\Lambda\sqrt{\mathrm{d}l^2-\mathrm{d}s^2}=\tfrac{1}{2}\int_0^\Lambda\mathrm{d}L \tag{14}$$

and $\mathrm{d}L=\sqrt{1-(v_{\mathcal{P}}^2/u_{\mathcal{P}}^2)}\,\mathrm{d}l$ is called an infinitesimal contracted length of the curve because the speed of transportation $\mathrm{d}s/\mathrm{d}\lambda=v_{\mathcal{P}}$ is always less than that of the rate of change of arc length of the curve; we call $\sqrt{1-v_{\mathcal{P}}^2/u_{\mathcal{P}}^2}$ it the contraction factor (CF). Equation (14) is an important result, since it reflects in a straightforward way certain features of the quantum state space. It also gives a novel way of looking into the geometric phase. Since 'length' and 'distance' dictate the topology of the curve $\hat{C}$, and, the geometric phase depends on them, it is also topological in nature. We have claimed in [16] that our expression is a more geometric one than other expressions previously known for time evolution of the quantum system. But here we have generalized it to an arbitrary parametric evolution of the quantum system, therefore the earlier result may be considered as a special case of this evolution.

An important observation made in this context is that if the evolution happens to be such that the contraction factor goes to zero, then obviously the geometric phase is zero. However, the length of the curve and the geometric distance function are non-zero. This is possible during the evolution if the quantum system passes through the shortest geodesic, then the length of the curve is minimized and is equal to the distance and consequently the contraction factor vanishes. Thus one recovers the result that for shortest geodesic joining initial and final points, the geometric phase vanishes [4]. If the contraction factor approaches unity, the geometric phase acquired by the system is just equal to half of the total length of the curve during the cyclic excursion. Also, it provides an important clue to the question: what will be the maximum geometric phase acquired during an arbitrary cyclic evolution? The answer is that it is half of the total length of the curve $l(C)$, in $\mathcal{P}$ (since the contraction factor varies from zero to unity). One more thing to be noted is that if the contraction factor is constant, then the geometric phase is directly proportional to the total length of the curve during the cyclic evolution.

In addition to this, (14) provides a novel way of understanding the origin of the geometric phase. At each point, λ the 'length' is greater than the 'distance'; due to this fundamental inequality between two geometric objects, another geometric object develops and it is recognized as the geometric phase. Thus, during the cyclic quantum evolution, we may regard the excess length of the curve over the distance to go on accumulating, so that its integrated, squared difference finally appears as the geometric phase. We emphasize here that for this interpretation to be valid, the generator of the parametric evolution, A need not be Hermitian or linear and it may or may not depend on the parameter λ. The simple expression (14) gives every answer to the question that we want to ask about the geometric phase.

Furthermore, our result provides a simple way of calculating the geometric phase for arbitrary cyclic, parametric evolution of quantum systems. We illustrate this here by studying one example. It differs from other known methods of calculating the geometric phase, because we do it via the geometric distance function and length of the curve and thereby presents a tractable algorithm for calculating the geometric phase.

The example comprises of a quantum system where the state vector is represented by

$$|\psi(\lambda)\rangle = (2J+1)^{-1/2} \sum_{n=-J}^{J} e^{-ina\lambda}|n\rangle \tag{15}$$

for $n = -J, -J+1, \ldots, J-1, J$, where the $|n\rangle$'s are fixed orthonormal states and a is a positive, real constant. The state has $(2J+1)$ discrete quantum levels, where J is a positive integer or half-odd integer. For simplicity of the calculation we have assumed $|\psi\rangle$ is normalized, but it does not affect the method of calculation in any way. The generator of the parameter satisfies the equation

$$A|n\rangle = na|n\rangle. \tag{16}$$

Because of (16), we can see that $|\psi(\lambda)\rangle$ obeys the equation $\langle\psi(\lambda)|\mathrm{d}/\mathrm{d}\lambda|\psi(\lambda)\rangle = 0$, i.e. it undergoes parallel transport with respect to the natural connection defined on $\mathscr{P}$. The holonomy associated with this connection determines the change of the phase of the state vector along a closed curve $\hat{C}$ in $\mathscr{P}$. This $|\psi(\lambda)\rangle$ is also called the horizontal lift of a closed curve $\hat{C}$ in $\mathscr{P}$ to which they project. In the interval $[0, \Lambda]$ if it undergoes a cyclic evolution, then

$$|\psi(\Lambda)\rangle = e^{i\phi}|\psi(0)\rangle = e^{i\beta}|\psi(0)\rangle \tag{17}$$

where β is the holonomy transformation associated with the curve $\hat{C}$ and is the geometric phase, which is again the same as the total phase acquired during a cyclic evolution. The system passes through a sequence of orthonormal state vectors; assuming the J's are half-odd integers we can see that the state changes sign after a period of $\Lambda = 2\pi/a$. Thus the total phase can be taken as π.

Now we can calculate the geometric distance function ds. It follows trivially from the definition and is given by

$$\mathrm{d}s = 2\frac{a}{\sqrt{3}}(J^2+J)^{1/2}\,\mathrm{d}\lambda. \tag{18}$$

Hence, during the cyclic evolution, the total distance travelled by the state vector $|\psi\rangle$ and measured by the Fubini–Study metric is given by

$$s = 4\frac{\pi}{\sqrt{3}}(J^2+J)^{1/2}. \tag{19}$$

To calculate the geometric length of the curve, we have to choose a single valued state vector $|\tilde{\psi}(\lambda)\rangle$ satisfying $|\tilde{\psi}(\Lambda)\rangle = |\tilde{\psi}(0)\rangle$. This is given by

$$|\tilde{\psi}(\lambda)\rangle = (2J+1)^{-1/2} \sum_{n=-J}^{J} e^{-i(n+1/2)a\lambda}|n\rangle. \tag{20}$$

We calculate the infinitesimal (squared) length of the curve,

$$dl^2 = 4[\tfrac{1}{3}(J^2+J)+\tfrac{1}{4}]a^2\,d\lambda^2. \tag{21}$$

The rate of change of the arc length of the curve is given by

$$u_{\mathcal{P}} = 2[\tfrac{1}{3}(J^2+J)+\tfrac{1}{4}]^{1/2}a$$

and the total length of the curve duirng one cycle is given by

$$l(c) = 4\pi[\tfrac{1}{3}(J^2+J)+\tfrac{1}{4}]^{1/2}. \tag{22}$$

Next, using (14), we calculate the geometric phase,

$$\beta = \tfrac{1}{2}\int_0^{\Lambda}\sqrt{dl^2 - ds^2} = \pi \tag{23}$$

which is exactly what was expected. Hence the geometric phase β, associated with the cyclic evolution of the parametric state, is π over one period $\Lambda = 2\pi/a$.

This example provides a geometric way of realizing the calculation of the phase during a cyclic excursion. One can also see the geometric nature of the three objects in $\mathcal{P}$, i.e. the phase, distance and length all as functions of a, J and Λ. An important quantity of interest in this context is the contraction factor (CF) i.e. $\sqrt{1-(v_{\mathcal{P}}^2/u_{\mathcal{P}}^2)}$, which is found to be $\sqrt{3/[4J(J+1)+3]}$. A necessary and sufficient condition for acquiring the geometric phase is that CF should not vanish during a cyclic evolution of the quantum system. Here in fact it is non-zero, unless J is very large, i.e. $J \to \infty$. Also we cannot have CF equal to unity (because the J's are half-odd integers) and hence the geometric phase cannot be made equal to the length of the curve during a cyclic evolution. This confirms our assertion that the geometric phase is equal to (half) the integral of the contracted length of the curve during cyclic evolution of the quantum system.

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